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AN APPLICATION OF THE SUBORDINATION CHAINS

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Abstract

The notion of differential superordination was introduced in [4] by S.S. Miller and P.T. Mocanu as a dual concept of differential subordination [3] and was developed in [5]. The notion of strong differential subordination was introduced by J.A. Antonino and S. Romaguera in [1]. In [6] the author introduced the dual concept of strong differential superordination. In this paper we study strong differential superordination using the subordination chains.

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1. Introduction and preliminaries

Let U denote the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}$$

and

$$\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}.$$

Let $\mathcal{H}(U\times\overline{U})$ denote the class of analytic functions in $U\times\overline{U}$. In [7] the authors define the classes

$$\mathcal{H}^*[a,n,\xi]=\{f\in\mathcal{H}(U\times\overline{U}):\ f(z,\xi)=a+a_n(\xi)z^n$$
$$+a_{n+1}(\xi)z^{n+1}+\dots,\ z\in U,\ \xi\in\overline{U}\},$$

with $a_k(\xi)$ holomorphic functions in $\overline{U}$, $k\geq n$,

$$\mathcal{H}_u(U)=\{f\in\mathcal{H}^*[a,n,\xi]:\ f(\cdot,\xi)\text{ univalent in }U\text{ for all }\xi\in\overline{U}\},$$

and let

$$K=\left\{f\in\mathcal{H}^*[a,n,\xi]:\ Re\,\frac{zf''(z,\xi)}{f'(z,\xi)}+1>0,\ z\in U\text{ for all }\xi\in\overline{U}\right\}$$

the class of convex functions,

$$S^*=\{f\in\mathcal{H}^*[a,n,\xi]:\ Re,\frac{zf'(z,\xi)}{f(z,\xi)}>0,\ z\in U\text{ for all }\xi\in\overline{U}\}$$

the class of starlike functions.

In order to prove our main results we use the following definitions and lemma:

Definition 1. ([7]) Let $h(z,\xi)$, $f(z,\xi)$ be analytic functions in $U\times\overline{U}$. The function $f(z,\xi)$ is said to be strongly subordinate to $h(z,\xi)$, or $h(z,\xi)$ is said to be strongly superordinate to $f(z,\xi)$, if there exists a function w analytic in U, with $w(0)=0$, $|w(z)|<1$ such that

$$f(z,\xi)=h(w(z),\xi),\ \text{for all }\xi\in\overline{U},\quad z\in U.$$

In such a case we write

$$f(z,\xi)\prec\prec h(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

Remark 1. (i) If $f(z,\xi)$ is analytic in $U\times\overline{U}$ and univalent in U for all $\xi\in\overline{U}$, Definition 1 is equivalent to

$$h(0,\xi)=f(0,\xi)\text{ for all }\xi\in\overline{U}\text{ and }h(U\times\overline{U})\subset f(U\times\overline{U}).$$

(ii) If $h(z,\xi)\equiv h(z)$ and $f(z,\xi)\equiv f(z)$ then the strong superordination becomes the usual notion of superordination.

DEFINITION 2. ([7]) We denote by Q the set functions $q(\cdot,\xi)$ that are analytic and injective, as function of z on $\overline{U}\setminus E(q)$ where

$$E(q)=\left\{\zeta\in\partial U:\ \lim_{z\to\zeta}q(z,\xi)=\infty\right\}$$

and are such that $q'(\zeta,\xi)\neq 0$, for $\zeta\in\partial U\setminus E(q)$, $\xi\in\overline{U}$.

The subclass of Q for which $q(0,\xi)=a$ is denoted by $Q(a)$.

LEMMA 1. ([8, Th. 2]) *Let $h(\cdot,\xi)$ be analytic in $U\times\overline{U}$, $q(\cdot,\xi)\in\mathcal{H}^*[a,n,\xi]$, $\varphi\in\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$, and suppose that*

$$\varphi(q(z,\xi),tzq'(z,\xi);\zeta,\xi)\in h(U\times\overline{U}),$$

for $z\in U$, $\zeta\in\partial U$, $\xi\in\overline{U}$ and $0<t\leq\frac{1}{n}\leq 1$. If $p(\cdot,\xi)\in Q(a)$ and $\varphi(p(z,\xi),zp'(z,\xi);z,\xi)$ is univalent in U, for all $\xi\in\overline{U}$, then

$$h(z,\xi)\prec\prec\varphi(p(z,\xi),zp'(z,\xi);z,\xi),\quad z\in U,\ \xi\in\overline{U}$$

implies

$$q(z,\xi)\prec\prec p(z,\xi),\quad z\in U,\ \xi\in\overline{U}.$$

Furthermore, if

$$\varphi(q(z,\xi),zq'(z,\xi);z,\xi)=h(z,\xi),\quad z\in U,\ \xi\in\overline{U}$$

has a univalent solution $q(\cdot,\xi)\in Q(a)$, then $q(\cdot,\xi)$ is the best subordinant, for all $\xi\in\overline{U}$.

DEFINITION 3. ([6]) Let $\varphi:\mathbb{C}^2\times U\times\overline{U}\to\mathbb{C}$ and h be analytic in $U\times\overline{U}$.

If $p(\cdot,\xi)\in\mathcal{H}^*[a,n,\xi]$ and $\varphi[p(z,\xi),zp'(z,\xi);z,\xi)$ are univalent in U, for all $\xi\in\overline{U}$ and satisfy the (first-order) strong differential superordination

$$h(z,\xi)\prec\prec\varphi(p(z,\xi),zp'(z,\xi);z,\xi),\quad z\in U,\ \xi\in\overline{U},\tag{1}$$

then $p(\cdot,\xi)$ is called a solution of the strong differential superordination. An analytic function $q(\cdot,\xi)$ is called a subordinant of the solutions of the strong differential superordination, or simply a subordinant if $q(z,\xi)\prec\prec p(z,\xi)$ for all $\xi\in\overline{U}$, for all $p(\cdot,\xi)$ satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q(z,\xi)\prec\prec\widetilde{q}(z,\xi)$ for all $\xi\in\overline{U}$, for all subordinants $q(\cdot,\xi)$ of (1) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of U.

2. Main results

Using the definitions given by Pommerenke [9, p.157] and Miller Mocanu [5, p. 4], we introduce the following definition:

Definition 4. The function $L : U \times \overline{U} \times [0,\infty) \to \mathbb{C}$ is a strong subordination (or a Loewner) chain if $L(z,\xi;t)$ is analytic and univalent in U for $\xi \in \overline{U}$, $t \geq 0$, $L(z,\xi;t)$ is continuously differentiable on $\mathbb{R}^+$ for all $z \in U$, $\xi \in \overline{U}$, and $L(z,\xi;s) \prec\prec L(z,\xi;t)$ where $0 \leq s \leq t$.

The following lemma provides a sufficient condition for $L(z,\xi;t)$ to be a strong subordination chain and it was obtained following a result given in [2, Lemma 1.2.5].

Lemma 2. *The function*

$$L(z,\xi;t) = a_1(\xi,t)z + a_2(\xi,t)z^2 + \dots,$$

with $a_1(\xi,t) \neq 0$ *for* $\xi \in \overline{U}$, $t \geq 0$ *and* $\lim\limits_{t\to\infty} |a_1(\xi,t)| = \infty$, *is a strong subordination chain if*

$$Re z \frac{\partial L(z,\xi;t)/\partial z}{\partial L(z,\xi;t)/\partial t} > 0, \quad z \in U,\ \xi \in \overline{U},\ t \geq 0.$$

Let $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ be an analytic function in a domain $D \subset \mathbb{C}^2$, let $p(\cdot,\xi) \in \mathcal{H}(U \times \overline{U})$ such that $\varphi(p(z,\xi), zp'(z,\xi); z,\xi)$ is univalent in U for all $\xi \in \overline{U}$ and suppose that $p(\cdot,\xi)$ satisfies the first-order strong differential superordination

$$h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi); z,\xi), \quad z \in U,\ \xi \in \overline{U}. \tag{2}$$

In the case when

$$\varphi(p(z,\xi), zp'(z,\xi); z,\xi) = \alpha(p(z,\xi)) + \beta(p(z,\xi))\gamma(zp'(z,\xi)) \tag{3}$$

we determine conditions on h, α, β and γ so that the strong superordination (2) implies $q(z,\xi) \prec\prec p(z,\xi)$, $z \in U$, $\xi \in \overline{U}$, where $q(\cdot,\xi)$ is the largest function so that $q(z,\xi) \prec\prec p(z,\xi)$, $z \in U$, $\xi \in \overline{U}$ for all functions $p(\cdot,\xi)$ satisfying the first-order differential superordination (2), i.e. $q(\cdot,\xi)$ is the best subordinant.

THEOREM 1. *Let* $q(\cdot,\xi) \in \mathcal{H}^*[a,1,\xi]$, *let* $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$ *and let* $\varphi(q(z,\xi), zq'(z,\xi)) \equiv h(z,\xi)$, $z \in U$, $\xi \in \overline{U}$.

If $L(z,\xi;t) = \varphi(q(z,\xi), tzq'(z,\xi))$ *is a strong subordination chain, and* $p \in \mathcal{H}^*[a,1,\xi] \cap Q$, *then*

$$h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi)) \tag{4}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

Furthermore, if $\varphi(q(z,\xi), zq'(z,\xi)) = h(z,\xi)$ *has an univalent solution* $q(\cdot,\xi) \in Q$, *then* $q(\cdot,\xi)$ *is the best subordinant.*

P r o o f. Since $L(z,\xi,;t)$ is a strong subordination chain, we have

$$L(z,\xi;t) \prec\prec L(z,\xi;1),\ \text{for}\ z \in U,\ \xi \in \overline{U},\ 0 < t \le \frac{1}{n} \le 1,$$

or equivalently

$$\varphi(q(z,\xi), tzq'(z,\xi)) \prec\prec \varphi(q(z,\xi), zq'(z,\xi)) = h(z,\xi). \tag{5}$$

Since (5) implies $\varphi(q(z,\xi), tzq'(z,\xi)) \in h(U \times \overline{U})$ and using Lemma 1, we have

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $q(z,\xi)$ is the best subordinant. ■

THEOREM 2. *Let* $q(\cdot,\xi)$ *be a convex (univalent) function in the unit disc* U, *for all* $\xi \in \overline{U}$. *Let* $\alpha, \beta \in \mathcal{H}(D)$, *where* $D \supset q(U \times \overline{U})$ *is a domain, and let* $\gamma \in \mathcal{H}(\mathbb{C})$ *suppose that*

$$\operatorname{Re} \frac{\alpha'(q(z,\xi)) + \beta'(q(z,\xi))\gamma(tzq'(z,\xi))}{\beta(q(z,\xi))\gamma'(tzq'(z,\xi))} > 0, \tag{6}$$

$\forall\ z \in U,\ \xi \in \overline{U}$ and $\forall\ t \ge 0$.

If $p(\cdot,\xi) \in \mathcal{H}^*[q(0,\xi),1,\xi] \cap Q$, *with* $p(U \times \overline{U}) \subset D$ *and* $\alpha(p(z,\xi)) + \beta(p(z,\xi))\gamma(zp'(z,\xi))$ *is univalent in* U, *for all* $\xi \in \overline{U}$, *then*

$$\begin{aligned}\alpha(q(z,\xi)) + \beta(q(z,\xi))\gamma(zq'(z,\xi)) &= h(z,\xi) \\ &\prec\prec \alpha(p(z,\xi)) + \beta(p(z,\xi))\gamma(zp'(z,\xi))\end{aligned} \tag{7}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $q(\cdot,\xi)$ *is the best subordinant.*

P r o o f. Let $\varphi(p(z,\xi), zp'(z,\xi) = \alpha(p(z,\xi)) + \beta(p(z,\xi))\gamma(zp'(z,\xi))$. By the hypothesis, we have

$$h(z,\xi) \prec\prec \varphi(p(z,\xi), zp'(z,\xi)), \quad z \in U,\ \xi \in \overline{U} \tag{8}$$

and $\varphi(p(z,\xi), zp'(z,\xi))$ is univalent in U for all $\xi \in \overline{U}$.

If we let

$$L(z,t) = \alpha(q(z,\xi)) + \beta(q(z,\xi))\gamma(tzq'(z,\xi)) = a_1(t,\xi)z + a_2(t,\xi)z^2 + \ldots, \tag{9}$$

differentiating (9), we obtain

$$\frac{\partial L(z,t)}{\partial z} = \alpha'(q(z,\xi))\frac{\partial q(z,\xi)}{\partial z} \tag{10}$$

$$+\beta'(q(z,\xi))\frac{\partial q(z,\xi)}{\partial z}\gamma(tzq'(z,\xi)$$

$$+\beta(q(z,\xi))\gamma'(tzq'(z,\xi))\left[tq'(z,\xi) + tz\frac{\partial^2 q(z,\xi)}{\partial z^2}\right].$$

For $z = 0$, using (10), we have

$$\frac{\partial L(0,t)}{\partial z} = \alpha'(q(0,\xi))q'(0,\xi) + \beta'(q(0,\xi))q'(0,\xi)\gamma(0) \tag{11}$$

$$+\beta(q(0,\xi))\gamma'(0)tq'(0,\xi)$$

$$= \beta(q(0,\xi))\gamma'(0)q'(0,\xi)\left[t + \frac{\alpha'(q(0,\xi)) + \beta'(q(0,\xi))\gamma(0)}{\beta q(0,\xi)\gamma'(0)}\right].$$

From the univalence of q we have $q'(0,\xi) \neq 0$ and by using (6) for $z = 0$ we deduce that

$$\frac{\partial L(0,t)}{\partial z} = a_1(t,\xi) \neq 0,\ \forall\ t \geq 0 \tag{12}$$

and

$$\lim_{t\to\infty} |a_1(t,\xi)| = \infty.$$

We calculate

$$\frac{\partial L(z,t)}{\partial t} = \beta(q(z,\xi))\gamma'(tzq'(z,\xi))zq'(z,\xi) = a_1'(t,\xi)z + a_2'(t,\xi)z^2 + \ldots$$

A simple calculus shows that

$$\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} = \frac{z\alpha'(q(z,\xi))q'(z,\xi) + z\beta'(q(z,\xi))q'(z,\xi)\gamma(tzq'(z,\xi))}{\beta(q(z,\xi))\gamma'(tzq'(z,\xi))zq'(z,\xi)} \tag{13}$$

$$+\frac{z\beta(q(z,\xi))\gamma'(tzq'(z,\xi))[tq'(z,\xi)+tzq''(z,\xi)]}{\beta(q(z,\xi))\gamma'(tzq'(z,\xi))zq'(z,\xi)}$$

$$=\frac{\alpha'(q(z,\xi))+\beta'(z,\xi)\gamma(tzq'(z,\xi))}{\beta(q(z,\xi))\gamma'(tzq'(z,\xi))}+t\left[1+\frac{zq''(z,\xi)}{q'(z,\xi)}\right].$$

We evaluate

$$Re\,\frac{z\partial L(z,\xi)/\partial z}{\partial L(z,\xi)/\partial t} \tag{14}$$

$$=\mathrm{Re}\,\left\{\frac{\alpha' q(z,\xi)+\beta'(z,\xi)\gamma(tzq'(z,\xi))}{\beta q(z,\xi)\gamma'(tzq'(z,\xi))}+t\left[1+\frac{zq''(z,\xi)}{q'(z,\xi)}\right]\right\}.$$

According to (6) and using the fact that $q(\cdot,\xi)$ is a convex function in U for all $\xi\in\overline{U}$, we obtain

$$Re\,\left[\frac{z\partial L(z,\xi)/\partial z}{\partial L(z,\xi)/\partial t}\right]>0,\quad z\in U,\ \xi\in\overline{U},\ t\geq 0, \tag{15}$$

and by Lemma 2 we conclude that L is a subordination chain. Now, applying Lemma 1, we obtain

$$q(z,\xi)\prec\prec p(z,\xi),\quad z\in U,\ \xi\in\overline{U},$$

and $q(\cdot,\xi)$ is the best subordinant.

Taking $\beta(w)\equiv 1$ in the above theorem we get the next corollary. ∎

Corollary 1. *Let $q(\cdot,\xi)$ be a convex (univalent) function in U for all $\xi\in\overline{U}$, $\alpha\in\mathcal{H}(D)$, where $D\supset q(U\times\overline{U})$ is a domain, and let $\gamma\in\mathcal{H}(\mathbb{C})$. Suppose that*

$$Re\,\frac{\alpha'(q(z,\xi))}{\gamma'(tzq'(z))}>0,\ \forall\ z\in U,\ \xi\in\overline{U}\text{ and }\forall\ t\geq 0.$$

If $p(\cdot,\xi)\in\mathcal{H}^*[q(0,\xi),1,\xi]\cap Q$, with $p(U\times\overline{U})\subset D$, and $\alpha(p(z,\xi))+\gamma(zp'(z,\xi))$ is univalent in U, for all $\xi\in\overline{U}$, then

$$\alpha(q(z,\xi))+\gamma(zq'(z,\xi))\prec\prec\alpha(p(z,\xi))+\gamma(zp'(z,\xi))$$

implies

$$q(z,\xi)\prec\prec p(z,\xi),\quad z\in U,\ \xi\in\overline{U}$$

and $q(\cdot,\xi)$ is the best subordinant.

For the particular case when $\gamma(w)=w$, Theorem 2 can be rewritten as follows:

Corollary 2. *Let $q(z,\xi)$ be a univalent function in U for all $\xi \in \overline{U}$ and let $\alpha, \beta \in \mathcal{H}(D)$, where $D \supset q(U \times \overline{U})$ is a domain. Suppose that:*

(i) Re $\dfrac{\alpha'(q(z,\xi))}{\beta(q(z,\xi))} > 0, \ \forall\ z \in U, \ \xi \in \overline{U}$ *and*

(ii) $Q(z,\xi) = zq'(z,\xi)\beta(q(z,\xi))$ *is a starlike (univalent) function in U for all $\xi \in \overline{U}$.*

If $p(\cdot,\xi) \in \mathcal{H}^*[q(0,\xi),1] \cap Q$, with $p(U \times \overline{U}) \subset D$, and $\alpha(p(z,\xi)) + zp'(z,\xi)\beta(p(z,\xi))$ is univalent in U for all $\xi \in \overline{U}$, then

$$\alpha(q(z,\xi)) + zq'(z,\xi)\beta(q(z,\xi)) \prec\prec \alpha(p(z,\xi)) + zp'(z,\xi)\beta(p(z,\xi))$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U, \ \xi \in \overline{U}$$

and $q(\cdot,\xi)$ is the best subordinant.

For the case $\beta(w) = 1$, using the fact that the function $Q(z,\xi) = zq'(z,\xi)$ is starlike (univalent) in U for all $\xi \in \overline{U}$ if and only if $q(\cdot,\xi)$ is convex (univalent) in U for all $\xi \in \overline{U}$, Corollary 2 becomes:

Corollary 3. *Let $q(\cdot,\xi)$ be a convex (univalent) function in U for all $\xi \in \overline{U}$ and let $\alpha \in \mathcal{H}(D)$, where $D \supset q(U \times \overline{U})$ is a domain.*

Suppose that

$$Re\alpha'(q(z,\xi)) > 0, \quad z \in U, \ \xi \in \overline{U}. \tag{16}$$

If $p(\cdot,\xi) \in \mathcal{H}^*[q(0,\xi),1,\xi] \cap Q$, with $p(U \times \overline{U}) \subset D$ and $\alpha(p(z,\xi)) + zp'(z,\xi)$ is univalent in U for all $\xi \in \overline{U}$, then

$$\alpha(q(z,\xi)) + zq'(z,\xi) \prec\prec \alpha(p(z,\xi)) + zp'(z,\xi)$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U, \ \xi \in \overline{U},$$

and $q(\cdot,\xi)$ is the best subordinant.

Next we will give some particular cases of the above results obtained for appropriate choices of the $q(\cdot,\xi)$, α and β functions.

Example 1. Let $q(\cdot,\xi)$ be a convex (univalent) function in U for all $\xi \in \overline{U}$ and let $\gamma \in \mathbb{C}$, with $Re\gamma > 0$. If $p(\cdot,\xi) \in \mathcal{H}^*[p(0,\xi),1,\xi] \cap Q$ and $p(z,\xi) + \frac{zp'(z,\xi)}{\gamma}$ is univalent in U for all $\xi \in \overline{U}$, then

$$q(z,\xi) + \frac{zq'(z,\xi)}{\gamma} \prec\prec p(z,\xi) + \frac{zp'(z,\xi)}{\gamma}$$

implies

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $q(\cdot,\xi)$ is the best subordinant.

P r o o f. Taking $\alpha(w) = w$ and $\beta(w) = \frac{1}{\gamma}$, $Re\,\gamma > 0$, in Corollary 2, condition (i) holds if $Re\gamma > 0$ and (ii) holds if and only if $g(\cdot,\xi)$ is a convex (univalent) function in U for all $\xi \in \overline{U}$. From Corollary 2, we have

$$q(z,\xi) \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}.$$

EXAMPLE 2. Let $\beta > 0$, $q(\cdot,\xi)$ be a univalent function in U for all $\xi \in \overline{U}$ and suppose that

$$Re\, q(z,\xi) > \beta, \quad z \in U. \tag{17}$$

If $p(\cdot,\xi) \in \mathcal{H}^*[q(0,\xi),1] \cap Q$ and $\frac{p^2(z,\xi)}{2} - \beta p(z,\xi) + zp'(z,\xi)$ is univalent in U for all $\xi \in \overline{U}$, then

$$\frac{q^2(z,\xi)}{2} - \beta q(z,\xi) + zq'(z,\xi) \prec\prec \frac{p^2(z,\xi)}{2} - \beta p(z,\xi) + zp'(z,\xi)$$

implies $q(z,\xi) \prec\prec p(z,\xi)$, for all $\xi \in \overline{U}$ and $q(\cdot,\xi)$ is the best subordinant.

P r o o f. If we consider in Corollary 3 the case

$$\alpha(w) = \frac{w^2}{2} - \beta w,$$

then we may easily see that (16) is equivalent to (17).

From Corollary 3, we have

$$q(z,\xi) \prec\prec p(z,\xi),$$

and $q(\cdot,\xi)$ is the best subordinant.

REMARK 2. The function

$$q(z,\xi) = \frac{\xi + (2\beta - 1)z\xi}{1+z}, \quad 0 < \beta < \frac{1}{Re\,\xi},\ \xi \in \overline{U},\ z \in U,$$

is convex (univalent) in U for all $\xi \in \overline{U}$ and $Re\, q(z,\xi) > \beta Re\,\xi$, $z \in U$, $\xi \in \overline{U}$. Hence, by using Example 2 we have:

If $p(\cdot,\xi) \in \mathcal{H}\left[\frac{\xi}{2}, 1, \xi\right] \cap Q$ such that $\frac{p^2(z,\xi)}{2} - \beta p(z,\xi) + zp'(z,\xi)$ is univalent in U and $\beta < \frac{1}{\operatorname{Re}\xi}$, then

$$\frac{\xi^2[1+2(2\beta-1)z+(2\beta-1)^2z^2]}{2(1+z)^2} - \beta \cdot \frac{\xi[1+(2\beta-1)z]}{1+z} + z \cdot \frac{\xi(2\beta-2)}{(1+z)^2}$$

$$\prec\prec \frac{p^2(z,\xi)}{2} - \beta p(z,\xi) + zp'(z,\xi)$$

implies

$$\frac{\xi + (2\beta-1)z\xi}{1+z} \prec\prec p(z,\xi), \quad z \in U,\ \xi \in \overline{U}$$

and $\frac{\xi+(2\beta-1)z\xi}{1+z}$ is the best subordinant.

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